arXiv:hep-th/0504142v1 18 Apr 2005

hep-th/0504142
UCLA-05-TEP-13

Black Rings in Taub-NUT

Iosif Bena[(1)], Per Kraus[(1)], and Nicholas P. Warner[(2)]

[(1)] *Department of Physics and Astronomy*
University of California
Los Angeles, CA 90095, USA

[(2)] *Department of Physics and Astronomy*
University of Southern California
Los Angeles, CA 90089-0484, USA

We construct the most generic three-charge, three-dipole-charge, BPS black-ring solutions in a Taub-NUT background. These solutions depend on seven charges and six moduli, and interpolate between a four-dimensional black hole and a five-dimensional black ring. They are also instrumental in determining the correct microscopic description of the five-dimensional BPS black rings.

1. Introduction

A new window into black-hole physics has opened up recently with the prediction [1,2] and subsequent discovery [3,4,5,6] of supergravity solutions for BPS black rings and three charge supertubes. Among other things, these solutions provide counterexamples to black-hole uniqueness, allow for a more detailed map between bulk and boundary states in the AdS/CFT correspondence, and lead to a generalized version of the black hole attractor mechanism [7]. Undoubtedly, much more remains to be discovered.

Our focus here is on presenting the details of a solution that two of the present authors conjectured in an earlier paper [8]. That paper gave a solution representing a two-charge supertube wrapped around the fibre of a Taub-NUT space. In the type IIB frame these solutions represent microstates of the D1-D5-KK system, and it was shown that the solutions are non-singular and in perfect accord with CFT expectations. Since the size of the Taub-NUT circle stabilizes at large radius, this solution is an asymptotically flat four dimensional solution carrying angular momentum. In the conclusion of [8] it was noted that the natural extension is to replace the two-charge supertube by a three-charge black ring, and that the corresponding solution would represent a four-dimensional BPS black hole carrying angular momentum. Such solutions have also been explored in [9].

Here we will present the general version of this class of solutions. It turns out to be quite straightforward to construct, either by solving directly the equations that give five-dimensional BPS three-charge solutions [10,4], or by using the general tools for finding BPS solutions with Gibbons-Hawking base spaces [6]. In particular, these solutions are specified by eight harmonic functions that can be chosen freely, subject to the constraints of smoothness and asymptotic flatness. The coefficients in the harmonic functions represent combinations of charges and moduli, whose interpretations we spell out. We also point out that the absence of closed timelike curves is governed by two functions, one of which is the $E_{7(7)}$ quartic invariant of the eight harmonic functions that specify the solution.

One of our aims is to resolve some of the confusion regarding the charges and the microscopic interpretation of black rings. Based on the D-brane physics underlying the existence of three charge supertubes and black rings [1], in [4] it was argued that the charge of BPS black-ring solutions has a component corresponding to charge dissolved into fluxes, and another component that comes from the local charge of the ring. In [11] it was then argued that one can give a microscopic description of the black ring entropy by realizing that the near-ring geometry is similar to the five-dimensional lift of a four-dimensional black hole with charges given by the local charges of the ring. The $E_{7(7)}$ invariant microscopic entropy derived from the CFT of this four-dimensional black hole correctly reproduces the black ring entropy.

However, these local charges differ from the black-ring charges measured at infinity, and in [12] it was subsequently proposed to base the CFT description of black rings on these asymptotic charges. Taking into account the zero point shift of the level number, these authors also reproduced the black ring entropy formula. The relation of these two approaches was, and remains, unclear. Moreover, in [13] it was argued that using a certain

mathematically correct definition of charge, one measures the same charge both at infinity and on the ring. This was then used to argue that the charges of the black ring are not the local charges introduced in [4], but the charges measured at infinity, and that black ring solutions have no charge dissolved in fluxes.

Knowing the solution for a black ring in Taub-NUT, as proposed in [8], will allow us to give a much cleaner derivation of the relation between the four-dimensional black hole and the five-dimensional black ring[1]. By adjusting moduli we can continuously tune the radius of the ring. For small radius the ring sits near the locally flat origin of Taub-NUT, and so reduces to a five-dimensional black ring. For large radius the ring sits in the region where the Taub-NUT circle stabilizes at fixed size, and so the solution becomes a four-dimensional black hole. The entropy depends only upon quantized charges, and so it is constant during the interpolation from small to large radius. One can then check in the large radius limit whether the microscopic charges of the ring, and its four-dimensional black-hole description, correspond to the local charges used in [4] and [11] or to the asymptotic charges used in [12]. We will see that it is the former, and we take this as evidence for the correctness of the logic of [11]. It remains an interesting question to explain the success of the alternative computation, since it seems unlikely to be an accident.

This interpolation also establishes that the microscopic charges of the five-dimensional BPS black ring are the local charges discussed in [4], and not the charges measured at infinity. Hence, the charge introduced in [13], while mathematically well-defined, does not measure the microscopic charge of the black rings, but some other quantity. The meaning of this quantity is clear when the black ring is in the near-tip region: it is the charge measured asymptotically in five dimensions. However, in the limit when the black ring is far away from the tip, in the four-dimensional region, the meaning of this quantity is much less clear.

The remainder of this paper is organized as follows. In section 2 we focus on explaining the black-ring entropy by adjusting the ring radius. We will do this in the context of the simplest version of the black ring in Taub-NUT, so as not to obscure the basic physics. In section 3 we present the form of the general three-charge solutions with a Gibbons-Hawking base. We re-derive the equations and form of the solution, and give a systematic method for solving these equations. In section 4 we use this to construct the general black ring in a Taub-NUT background. This solution contains a number of adjustable moduli, whose physical meaning we discuss.

Note: As this work was being completed, two papers [15,16] appeared which also discuss the black ring in Taub-NUT.

---

[1] Note that this type of approach was exploited recently in [14] to relate the five-dimensional BMPV black hole to four-dimensional black holes, and to thus give a five-dimensional interpretation to the topological string partition function.

2. Black ring in Taub-NUT

In this section we will give a physically motivated derivation of the simplest black ring in Taub-NUT, and show how adjusting the radius of the black ring allows one to interpolate between a black ring in five dimensions and a black hole in four dimensions.

Our starting point is the five-dimensional black ring presented in [3,4,5,6]. In this section we will only pay attention to the metric, which captures the basic physics of the problem, and defer discussion of the field strengths and moduli to the next section. The black ring metric is:

$$\begin{aligned} ds^2 &= -(Z_1Z_2Z_3)^{-2/3}(dt+k)^2 + (Z_1Z_2Z_3)^{1/3}\Big(d\tilde{r}^2 + \tilde{r}^2(d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\psi}^2 + \cos^2\tilde{\theta} d\tilde{\phi}^2)\Big) \\ Z_I &= 1 + \frac{\overline{Q}_I}{\tilde{\Sigma}} + \tfrac{1}{2}C_{IJK}q^Jq^K\frac{\tilde{r}^2}{\tilde{\Sigma}^2} \\ k &= -\frac{\tilde{r}^2}{2\tilde{\Sigma}^2}\left(q^I\overline{Q}_I + \frac{2q^1q^2q^3\tilde{r}^2}{\tilde{\Sigma}}\right)(\cos^2\tilde{\theta}\, d\tilde{\phi} + \sin^2\tilde{\theta}\, d\tilde{\psi}) - 3J_T\,\frac{2\tilde{r}^2\sin^2\tilde{\theta}}{\tilde{\Sigma}(\tilde{r}^2+\tilde{R}^2+\tilde{\Sigma})}\,d\tilde{\psi}\,, \end{aligned} \tag{2.1}$$

where $C_{IJK} = 1$ for $(IJK) = (123)$ and permutations thereof,

$$\tilde{\Sigma} = \sqrt{(\tilde{r}^2 - \tilde{R}^2)^2 + 4\tilde{R}^2\tilde{r}^2\cos^2\tilde{\theta}}\,, \tag{2.2}$$

and J_T is the difference between the two angular momenta of the ring: $J_T \equiv J_{\tilde{\psi}} - J_{\tilde{\phi}}$. The radius of the ring is $\tilde{R}$, and is related to J_T by

$$J_T = (q^1 + q^2 + q^3)\tilde{R}^2. \tag{2.3}$$

We wrote the solution in terms of the ring charges $\overline{Q}_I$. As we have noted, for the 5D black ring these differ from the charges measured at infinity, which are $Q_I = \overline{Q}_I + \frac{1}{2}C_{IJK}q^Jq^K$.

It is convenient to choose units such that $G_5 = \frac{\pi}{4}$, and choose the three T^2's that appear in the M-theory lift of this solution to have equal size. In these units, the charges Q_I, $\overline{Q}_I$ and q^I are quantized as integers.

We now perform a change of coordinates, to bring the black ring to a form in which we can easily include it in Taub-NUT. We define

$$\phi = \tilde{\phi} - \tilde{\psi}, \quad \psi = 2\tilde{\psi}, \quad \theta = 2\tilde{\theta}, \quad \rho = \frac{\tilde{r}^2}{4}\ . \tag{2.4}$$

The coordinate ranges are given by

$$\theta \in (0,\pi), \quad (\psi,\phi) \cong (\psi + 4\pi, \phi) \cong (\psi, \phi + 2\pi)\ . \tag{2.5}$$

In the new coordinates the black-ring metric is

$$
\begin{aligned}
ds^2 &= -(Z_1 Z_2 Z_3)^{-2/3}(dt+k)^2 + (Z_1 Z_2 Z_3)^{1/3} h_{mn} dx^m dx^n \,, \\
Z_I &= 1 + \frac{\overline{Q}_I}{4\Sigma} + \tfrac{1}{2} C_{IJK} q^J q^K \frac{\rho}{4\Sigma^2} \,, \\
k &= \mu \,(d\psi + (1+\cos\theta) d\phi) + \omega \,, \\
\mu &= -\frac{1}{16}\frac{\rho}{\Sigma^2}\left(q^I \overline{Q}_I + \frac{2q^1 q^2 q^3 \rho}{\Sigma}\right) + \frac{J_T}{16R}\left(1 - \frac{\rho}{\Sigma} - \frac{R}{\Sigma}\right) \,, \\
\omega &= -\frac{J_T \rho}{4\Sigma(\rho + R + \Sigma)} \sin^2\theta d\phi \,,
\end{aligned}
\tag{2.6}
$$

with

$$
\begin{aligned}
h_{mn} dx^m dx^n &= V^{-1}\,(d\psi + (1+\cos\theta)d\phi)^2 + V(d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)) \\
V &= \frac{1}{\rho}, \quad \Sigma = \sqrt{\rho^2 + R^2 + 2R\rho\cos\theta}, \quad R = \frac{\tilde{R}^2}{4} \,.
\end{aligned}
\tag{2.7}
$$

The first line of (2.7) is just flat space written in Gibbons-Hawking coordinates. In these coordinates, the ring is sitting at a distance R along the negative z axis of the three-dimensional base. To change the four-dimensional base metric into Taub-NUT one needs to add a 1 to the harmonic function V. This is easily accomplished using the general results of Gauntlett and Gutowski on solutions with Gibbons-Hawking base [6], which we also re-derive in the next section.

A general supersymmetric solution can be written in terms of harmonic functions $K^I,\ L_I,\ M$ and V as (our conventions differ from [6]):

$$
\begin{aligned}
Z_I &= \frac{1}{2} H^{-1} C_{IPQ} K^P K^Q + L_I \,, \\
\mu &= \frac{1}{6} V^{-2} C_{IPQ} K^I K^P K^Q + \frac{1}{2} V^{-1} L_I K^I + M \,, \\
\nabla \times \omega &= V \nabla M - M \nabla V + \frac{1}{2}(K^I \nabla L_I - L_I \nabla K^I) \,.
\end{aligned}
\tag{2.8}
$$

Indeed, we can check that the solution (2.6) takes this form with

$$
K^I = -\frac{q^I}{2\Sigma} \,, \qquad L_I = 1 + \frac{\overline{Q}_I}{4\Sigma} \,, \qquad M = \frac{J_T}{16}\left(\frac{1}{R} - \frac{1}{\Sigma}\right) \,, \qquad V = \frac{1}{\rho} \,. \tag{2.9}
$$

The virtue of this is that we can now modify the Gibbons-Hawking harmonic function to

$$
V = h + \frac{1}{\rho} \tag{2.10}
$$

for constant h and, using (2.8) and (2.9), we still have a solution. Actually, in order to avoid both Dirac string singularities and closed time-like curves, the relation (2.3) between J_T and the dipole charges must be modified to:

$$J_T\left(h+\frac{1}{R}\right)=4(q^1+q^2+q^3)\,. \tag{2.11}$$

This will be discussed in detail in later sections, but it follows because the absence of singularities in ω puts constraints on the sources in the third equation of (2.8).

For small ring radius, $R \ll 1$, or for small h, this reduces to the five-dimensional black ring described above. We now wish to consider the opposite limit, $R \gg 1$. However, if we want to keep the quantized charges of the ring fixed, the changing of R should be such that (2.11) remains satisfied. We can think of this as keeping the physical radius of the ring fixed while changing its position in Taub-NUT. In this limit the black ring is far from the Taub-NUT tip and effectively sees an infinite cylinder. In other words, it is physically clear that the black ring in this limit becomes a straight black string wrapped on a circle, which is nothing but a four-dimensional black hole.

To see this in more detail, we analyze the geometry in the region far from the tip, that is, for $\rho \gg 1$, where we can take $V = h$. We also want to center the three-dimensional spherical coordinates on the ring, and so we change to coordinates such that Σ is the radius away from the ring. We then have:

$$d\rho^2+\rho^2(d\theta^2+\sin^2\theta d\phi^2)=d\Sigma^2+\Sigma^2(d\hat{\theta}^2+\sin^2\hat{\theta}d\phi^2)\ , \tag{2.12}$$

and

$$\rho=\sqrt{\Sigma^2+R^2-2R\Sigma\cos\hat{\theta}},\quad \cos\theta=\frac{\Sigma\cos\hat{\theta}-R}{\rho}\ . \tag{2.13}$$

Taking $R\to\infty$, at fixed $(\Sigma,\hat{\theta},\hat{\phi})$ and $h+\frac{1}{R}$, we find that the metric is:

$$ds^2=-(\tilde{Z}_1\tilde{Z}_2\tilde{Z}_3)^{-2/3}(d\tilde{t}+\tilde{\mu}d\psi)^2+(\tilde{Z}_1\tilde{Z}_2\tilde{Z}_3)^{1/3}\left(dr^2+r^2(d\hat{\theta}^2+\sin^2\hat{\theta}d\hat{\phi}^2)\right)\ , \tag{2.14}$$

where

$$\tilde{Z}_I\equiv\frac{Z_I}{h}\,,\qquad \tilde{\mu}\equiv\frac{\mu}{h}\,,\qquad r\equiv h\Sigma\,,\qquad \tilde{t}\equiv\frac{t}{h}\,. \tag{2.15}$$

When written in terms of the coordinate r the harmonic functions become:

$$\tilde{Z}_I=\frac{1}{h}+\frac{\overline{Q}_I}{4r}+\frac{C_{IJK}q^Jq^K}{8r^2}\,,\qquad \tilde{\mu}=-\frac{J_T}{16r}-\frac{q^I\overline{Q}_I}{16r^2}-\frac{q^1q^2q^3}{8r^3}\,,\qquad \omega=0\,. \tag{2.16}$$

This is precisely the four-dimensional black hole found by wrapping the black string solution of [2] on a circle.

As noted in [11], the entropy of the 5D black ring takes a simple form in terms of the quartic invariant of $E_{7(7)}$, which gave the first clue to the relation with four-dimensional

black holes, since compactifications to four dimensions have an $E_{7(7)}$ duality group. The general entropy for this class of black holes is [17]:

$$S = 2\pi\sqrt{J_4}\ , \tag{2.17}$$

where J_4 is the quartic $E_{7(7)}$ invariant, which can be expressed in the basis (x_{ij}, y^{ij}) as

$$\begin{aligned} J_4 = &-\frac{1}{4}(x_{12}y^{12} + x_{34}y^{34} + x_{56}y^{56} + x_{78}y^{78})^2 - (x_{12}x_{34}x_{56}x_{78} + y^{12}y^{34}y^{56}y^{78}) \\ &+ x_{12}x_{34}y^{12}y^{34} + x_{12}x_{56}y^{12}y^{56} + x_{34}x_{56}y^{34}y^{56} + x_{12}x_{78}y^{12}y^{78} + x_{34}x_{78}y^{34}y^{78} \\ &+ x_{56}x_{78}y^{56}y^{78}. \end{aligned} \tag{2.18}$$

The black-ring entropy is recovered by taking

$$\begin{aligned} &x_{12} = \overline{Q}_1, \quad x_{34} = \overline{Q}_2, \quad x_{56} = \overline{Q}_3, \quad x_{78} = 0, \\ &y^{12} = q^1, \quad y^{34} = q^2, \quad y^{56} = q^3, \quad y^{78} = J_T = J_{\tilde{\psi}} - J_{\tilde{\phi}}. \end{aligned} \tag{2.19}$$

Hence, the "tube angular momentum" J_T plays the role of momentum in the four-dimensional black hole picture. As we explained above, for the asymptotically-flat 5D black ring, J_T is the difference of the two independent angular momenta, and is given by (2.3).

We now discuss the implications of this for a microscopic understanding of the BPS black ring. In terms of M-theory compactified on $T^6 \times S^1$ the solution (2.14) represents q^I M5-branes wrapped on the I'th 4-cycle, and $\overline{Q}_I$ M2-branes wrapped on the I'th 2-cycle. All the branes are also wrapped on S^1 with momentum J_T flowing on the intersection. The microscopic entropy for the most generic type of such a black hole has been computed in [18] based on the technology developed in [19][2].

Here we have shown that this entropy counting also applies to the five-dimensional black ring. The key point is that the entropy formula (2.17) is valid throughout the interpolation between the 5D black ring and the 4D black hole, as follows from the moduli independence of the entropy. This yields the same microscopic description of black rings as was presented in [11]. The difference is that there we had to rely on taking a near ring limit, which, while physically plausible, was not as rigorous as the Taub-NUT interpolation given here.

This analysis also shows that the microscopic charges and angular momentum of the five-dimensional black ring are not the same as the charge and angular momenta measured at infinity. This confirms the observation of [4] that the charge and angular momenta of the black-ring solution have a component coming from the microscopic charge and microscopic angular momentum of the ring, and another component coming from charge and angular

---

[2] It turns out that the analysis of [19] suffices for the particular combination of charges considered here.

momenta carried by the fluxes. This also shows that the charges defined in [13], while mathematically well defined, are not the microscopic charges of the black ring. It would be very interesting to find an interpretation of the charges of [13] in the limit when we are far away from the tip where the ring becomes a four-dimensional black hole. One puzzling feature is that as one puts together several four-dimensional black holes, of charges $\overline{Q}_{A,i}$ and q_i^A, the charges defined in [13], given by $Q_{A,i} = \overline{Q}_{A,i} + \frac{1}{2} C_{ABC} q_i^B q_i^C$, are not additive.

A few other issues relating to the entropy deserve comment. In [12] it was proposed to give the black ring a CFT description based on a four-dimensional black hole CFT with charges Q_I rather than $\overline{Q}_I$, and with momentum J_ψ rather than J_T. In order to recover the entropy formula (2.17) an important role was played by a non-extensive zero point energy shift of L_0. In light of the present understanding it is rather mysterious to us why this gives the right entropy, since we have shown explicitly that the relevant four-dimensional black hole CFT is the one with charges $\overline{Q}_I$, momentum J_T, and no zero point shift of L_0. We should also note that the microscopic description given here trivially carries over to the case of multi-ring solutions [20,6], yielding the correct entropy. On the other hand, the approach of [12] would seem to run into problems since the total charge Q_A is not simply a sum of the individual $Q_{A,i}$, but gets contributions from cross terms of the form $C_{ABC}\, q_i^B\, q_j^C$.

3. Three-charge solutions with Gibbons-Hawking base

3.1. The general system of equations

As shown in [10,4], an M-theory background that preserves the same supersymmetries as three orthogonal M2-branes can be written as:

$$\begin{aligned} ds_{11}^2 = &- \left(\frac{1}{Z_1 Z_2 Z_3} \right)^{2/3} (dt+k)^2 + (Z_1 Z_2 Z_3)^{1/3}\, h_{mn} dx^m dx^n \\ &+ \left(\frac{Z_2 Z_3}{Z_1^2} \right)^{1/3} (dx_1^2 + dx_2^2) + \left(\frac{Z_1 Z_3}{Z_2^2} \right)^{1/3} (dx_3^2 + dx_4^2) + \left(\frac{Z_1 Z_2}{Z_3^2} \right)^{1/3} (dx_5^2 + dx_6^2)\,, \end{aligned}$$

$$\mathcal{A} = A^1 \wedge dx_1 \wedge dx_2 + A^2 \wedge dx_3 \wedge dx_4 + A^3 \wedge dx_5 \wedge dx_6 \ , \tag{3.1}$$

where A^I and k are one-forms in the five-dimensional space transverse to the T^6. The metric h_{mn} is a four-dimensional hyper-Kähler metric.

When written in terms of the "dipole field strengths" Θ^I,

$$\Theta^I \equiv dA^I + d\Big(\frac{dt+k}{Z_I} \Big)\,, \tag{3.2}$$

the BPS equations simplify to [10,4]:

$$\begin{aligned} \Theta^I &= \star_4 \Theta^I \\ \nabla^2 Z_I &= \frac{1}{2} C_{IJK} \star_4 (\Theta^J \wedge \Theta^K) \\ dk + \star_4 dk &= Z_I \Theta^I \ , \end{aligned} \tag{3.3}$$

where $\star_4$ is the Hodge dual taken with respect to the four-dimensional metric h_{mn}. We will take the base to have a Gibbons-Hawking metric:

$$h_{mn} dx^m dx^n = V(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) + \frac{1}{V}\big(d\psi + \vec{A}\cdot d\vec{y}\big)^2 \tag{3.4}$$

with

$$\vec{\nabla} \times \vec{A} \ = \ \vec{\nabla} V. \tag{3.5}$$

Here we will, for the present, consider a completely general base with an arbitrary harmonic function, V. We will denote the one-form, $\vec{A}\cdot d\vec{y} \equiv A$.

This metric has the following orthonormal basis of 1-forms:

$$\hat{e}^1 \ = \ V^{-\frac{1}{2}}\,(d\psi \ + \ A)\,, \qquad \hat{e}^{a+1} \ = \ V^{\frac{1}{2}}\, dy^a\,, \quad a = 1,2,3\,. \tag{3.6}$$

There are two natural sets of two-forms:

$$\Omega_\pm^{(a)} \ \equiv \ \hat{e}^1 \wedge \hat{e}^{a+1} \ \pm \ \tfrac{1}{2}\,\epsilon_{abc}\,\hat{e}^{b+1} \wedge \hat{e}^{c+1}\,, \qquad a = 1,2,3\,. \tag{3.7}$$

The $\Omega_-^{(a)}$ are anti-self-dual and harmonic, defining the hyper- Kähler structure on the base. $\Omega_+^{(a)}$ are self-dual, and we can take the self-dual field strengths Θ^I to be proportional to them:

$$\Theta^I \ = \ -\sum_{a=1}^{3} \big(\partial_a\big(V^{-1}\,K^I\big)\big)\,\Omega_+^{(a)}\,. \tag{3.8}$$

For Θ^I to be closed, the functions K^I have to be harmonic in $\mathbb{R}^3$. Potentials satisfying $\Theta^I = dB^I$ are then:

$$B^I \ \equiv \ V^{-1}\,K^I\,(d\psi \ + \ A) \ + \ \vec{\xi}^I \cdot d\vec{y}\,, \tag{3.9}$$

where

$$\vec{\nabla} \times \vec{\xi}^I \ = \ -\vec{\nabla} K^I\,. \tag{3.10}$$

Hence, $\vec{\xi}^I$ are vector potentials for magnetic monopoles located at the poles of K^I.

The three self-dual Maxwell fields Θ^I are thus determined by the three harmonic functions K^I. Inserting this result in the right hand side of (3.3) we find:

$$Z_I \ = \ \tfrac{1}{2} C_{IJK} \frac{K^J K^K}{V} \ + \ L_I\,, \tag{3.11}$$

where L_I are three more independent harmonic functions.

We now write the one-form k as:

$$k \;=\; \mu(d\psi + A) \;+\; \omega \tag{3.12}$$

and then the last equation in (3.3) becomes:

$$\vec{\nabla} \times \vec{\omega} \;=\; (V\vec{\nabla}\mu \;-\; \mu\vec{\nabla}V) \;-\; V\sum_{i=1}^{3} Z_I\,\vec{\nabla}\Big(\frac{K^I}{V}\Big)\,. \tag{3.13}$$

Taking the divergence yields the following equation for μ:

$$\nabla^2\mu \;=\; 2\,V^{-1}\,\vec{\nabla}\cdot\Big(V\sum_{i=1}^{3} Z_I\;\vec{\nabla}\frac{K^I}{V}\Big)\,, \tag{3.14}$$

which is solved by:

$$\mu \;=\; \frac{1}{6}C_{IJK}\frac{K^IK^JK^K}{V^2} \;+\; \frac{1}{2\,V}K^IL_I \;+\; M\,, \tag{3.15}$$

where M is yet another harmonic function. Indeed, M determines the anti-self-dual part of dk that cancels out of the last equation in (3.3). Substituting this result for μ into (3.13) we find that ω satisfies

$$\vec{\nabla} \times \vec{\omega} \;=\; V\vec{\nabla}M - M\vec{\nabla}M + \frac{1}{2}(K^I\vec{\nabla}L_I - L_I\vec{\nabla}K^I)\,. \tag{3.16}$$

Summarizing, to construct a solution with three charges, three dipole charges, and a Gibbons-Hawking base one needs to specify *eight* harmonic functions in $\mathbb{R}^3$: V, K^I, L_I and M. The function V gives the Gibbons-Hawking base, the K^I determine the "dipole" charges, the L_I represent a contribution to the charges, and M determines the anti-self-dual part of dk.

The result of our analysis reproduces that of [21,6]. Here we have proceeded via the linear algorithm spelled out in [4]: first computing Θ^I and then using these as sources in the equations for Z_I. This procedure highlights an important "gauge invariance": we can add to K^I an arbitrary multiple of V without affecting Θ^I (which only depend on the derivatives of $\frac{K^I}{V}$). Hence, this change does not affect the solution. At the level of the equations, such a shift can be reabsorbed by shifting the L_I and M by appropriate harmonic functions, and does not affect the physical quantities Z_I, μ, ω, and Θ^I. The gauge invariance of the solutions can be written as:

$$\begin{aligned} K^I &\to K^I \;+\; l^I\,V\,, \\ L_I &\to L_I \;-\; C_{IJK}l^JK^K \;-\; \tfrac{1}{2}C_{IJK}l^Jl^KV\,, \\ M &\to M - \tfrac{1}{2}l^IL_I + \frac{1}{12}C_{IJK}\left(Vl^Il^Jl^K + 3l^Il^JK^K\right)\,, \end{aligned} \tag{3.17}$$

where the l^I are three arbitrary constants. This gauge invariance can be used to eliminate one of the terms that will appear in K^I, and greatly simplifies the solution. The system of equations above gives all solutions with a Gibbons-Hawking base.

3.2. Solving for ω

Since everything is determined by eight harmonic functions, all that remains is to solve for ω in equation (3.16). The right hand side has only terms of the form $H_1 \nabla H_2 - H_2 \nabla H_1$, where H_1 and H_2 are harmonic functions sourced at discrete points in the base, and possibly have an overall additive constant. Let $\vec{y}^{(j)}$ be the positions of the source points in the base, and let $r_j \equiv |\vec{y} - \vec{y}^{(j)}|$. The right-hand side of (3.16) therefore has terms of the form:

$$\frac{1}{r_i} \vec{\nabla} \frac{1}{r_j} \;-\; \frac{1}{r_j} \vec{\nabla} \frac{1}{r_i} \qquad \text{and} \qquad \vec{\nabla} \frac{1}{r_i} \,. \tag{3.18}$$

Hence ω will be built from the vectors $\vec{c}_{ij}$ and $\vec{v}_i$ satisfying

$$\vec{\nabla} \times \vec{c}_{ij} \;=\; \frac{1}{r_i} \vec{\nabla} \frac{1}{r_j} \;-\; \frac{1}{r_j} \vec{\nabla} \frac{1}{r_i} \qquad \text{and} \qquad \vec{\nabla} \times \vec{v}_i \;=\; \vec{\nabla} \frac{1}{r_i} \,. \tag{3.19}$$

To find $\vec{c}_{ij}$ and $\vec{v}_i$ one therefore only needs to consider pairs of source points, like $(\vec{y}^{(1)}, \vec{y}^{(2)})$. Choose coordinates so that $\vec{y}^{(1)} = (0,0,0)$ and $\vec{y}^{(2)} = (0,0,-R)$. Then the explicit solutions may be written very simply. Let $(y_1, y_2, y_3) = (x, y, z)$, $\Sigma = \sqrt{x^2 + y^2 + (z+R)^2}$, and let ϕ denote the polar angle in the (x, y)-plane. Then:

$$v_1 \;=\; \frac{z}{r}\, d\phi \,, \qquad v_2 \;=\; \frac{(z+R)}{\Sigma}\, d\phi \,, \tag{3.20}$$

and

$$c_{12} \;=\; -\frac{(r^2 + R\, z)}{R\, r\, \Sigma}\, d\phi \,. \tag{3.21}$$

One then converts these back to a more general system of coordinates and then adds up all the contributions to ω from all the pairs of points.

3.3. Absence of closed timelike curves

We have now obtained the general solution, but we still need to restrict our choice of harmonic functions to avoid unphysical pathologies. In particular, we consider the necessary conditions to avoid the appearance of closed timelike curves (CTCs).

If we define $W \equiv (Z_1 Z_2 Z_3)^{1/6}$, and use the expression for k in (3.12) then the metric along the time and four spatial directions of the base is

$$\begin{aligned} ds_5 = &- W^{-4} \big(dt + \mu (d\psi + A) + \omega \big)^2 + W^2 V^{-1} \left(d\psi + A \right)^2 \\ &+ W^2 V \big(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \big) \ . \end{aligned} \tag{3.22}$$

To look for CTC's in this metric, it is useful to examine the spatial components along

the Taub-NUT directions:

$$
\begin{aligned}
d\tilde{s}_4 \;=\;& -W^{-4}\left(\mu(d\psi+A)+\omega\right)^2 \\
&\qquad + \; W^2V^{-1}\big(d\psi+A\big)^2 + W^2V\big(dr^2+r^2d\theta^2+r^2\sin^2\theta\,d\phi^2\big) \\
=\;& W^{-4}\,(W^6V^{-1}-\mu^2)\Big(d\psi+A-\frac{\mu\,\omega}{W^6V^{-1}-\mu^2}\Big)^2 - \frac{W^2\,V^{-1}}{W^6V^{-1}-\mu^2}\,\omega^2 \\
&+\; W^2V\big(dr^2+r^2d\theta^2+r^2\sin^2\theta\,d\phi^2\big) \\
=\;& \frac{\mathcal{Q}}{W^4V^2}\Big(d\psi+A-\frac{\mu\,V^2}{\mathcal{Q}}\,\omega\Big)^2 + W^2V\Big(r^2\sin^2\theta\,d\phi^2-\frac{\omega^2}{\mathcal{Q}}\Big) + W^2V(dr^2+r^2d\theta^2)
\end{aligned}
\tag{3.23}
$$

where we have introduced

$$
\mathcal{Q} \equiv W^6V - \mu^2V^2 = Z_1Z_2Z_3V - \mu^2V^2 \ . \tag{3.24}
$$

Hence the closed timelike curves of the solution are controlled by two quantities: $\mathcal{Q}$ and $r^2\sin^2\theta d\phi^2 - \frac{\omega^2}{\mathcal{Q}}$, which must thus both be everywhere positive to avoid CTC's. In fact, upon evaluating $\mathcal{Q}$ as a function of the eight harmonic functions that determine the solution we reveal a beautiful result:

$$
\begin{aligned}
\mathcal{Q} = -M^2\,V^2 - \frac{1}{3}\,M\,C_{IJK}K^I\,K^J\,K^k - M\,V\,K^I\,L_I - \frac{1}{4}(K^IL_I)^2 \\
+\frac{1}{6}VC^{IJK}L_IL_JL_K + \frac{1}{4}C^{IJK}C_{IMN}L_JL_KK^MK^N
\end{aligned}
\tag{3.25}
$$

with $C^{IJK} = C_{IJK}$. We find that $\mathcal{Q}$ is nothing other than the $E_{7(7)}$ quartic invariant (2.18) where the x's are identified with $L_1, L_2, L_3, -V$, and the y's with $K_1, K_2, K_3, 2M$.

Positivity of $r^2\sin^2\theta\,d\phi^2 - \frac{\omega^2}{\mathcal{Q}}$ can be examined by looking near the axis between every pairs of points $y^{(i)}$ and $y^{(j)}$ described above. Here we will focus on the two-center solution, and use (3.20) and (3.21). Positivity implies that ω should vanish at $\sin\theta = 0$, that is, on the z-axis. For $R > 0$, the only combination of c_{12}, v_1, and v_2 that vanishes on the z-axis is:

$$
\begin{aligned}
\omega_0 \;&=\; (v_1 + R\,c_{12} - v_2 + d\phi) \\
&=\; -\frac{(x^2+y^2-(r+z)\,(z+R-\Sigma))}{r\,\Sigma}\,d\phi \ .
\end{aligned}
\tag{3.26}
$$

Hence, in order for the solution to have no closed timelike curves, ω must be proportional to ω_0.

Another obvious implication of the positivity of $\mathcal{Q}$ is that if we have no charges at the location of the singularities of V (*i.e.* Z_I are finite there), then μ at those locations must vanish.

4. The General Black Ring in Taub-NUT

We are now ready to construct the general three-charge, three-dipole-charge, BPS black ring on Taub-NUT. These rings are the generalization of the Taub-NUT supertube of [8].

To construct a black ring in Taub-NUT we take the harmonic function V to be sourced at the origin, while allowing the other harmonic functions to be sourced both at the origin and at the location of the ring. Some of the harmonic functions will also contain constants.

We take the Taub-NUT potential to be:

$$V \;=\; h \;+\; \frac{Q}{r}\,. \tag{4.1}$$

Allowing a general additive constant, h, is useful for interpolating between the five-dimensional and four-dimensional descriptions, as we saw in Section 2.

The other harmonic functions can be sourced both at the origin of Taub-NUT, and at the location of the ring, $\vec{y}^{(2)} \;=\; (0,0,-R)$. The harmonic functions that give the dipole charges are thus:

$$K^I \;=\; \frac{2\,\alpha^I}{r} \;+\; \frac{2\,\beta^I}{\Sigma} \;+\; 2\,\gamma^I\,. \tag{4.2}$$

To simplify this, we use the gauge invariance (3.17) to set $\alpha^I = 0$. The γ^I have a nice interpretation as the Wilson loops corresponding to the three gauge fields A^I wrapping the Taub-NUT circle at infinity, and we will keep them for completeness. Thus:

$$K^I \;=\; \frac{2\,\beta^I}{\Sigma} \;+\; 2\,\gamma^I\,. \tag{4.3}$$

By requiring that no charge be present at the origin of the Taub-NUT space we can set the coefficient of the r^{-1} term in the harmonic function L_I to be zero. Hence

$$L_I \;=\; \frac{g_I}{\Sigma} \;+\; c_I \;-\; \frac{2}{h} C_{IJK}\gamma^J\gamma^K\,. \tag{4.4}$$

The coefficient of the Σ^{-1} term contributes to the charge density of the ring. The constant was chosen such that c_I are the asymptotic values of the Z_I at infinity, which we allow to be arbitrary.

Since the Z_I are not sourced at the origin, $\mathcal{Q}$ can only be positive if μ vanishes at $r = 0$. This first implies that the coefficient of the r^{-1} term in the harmonic function M is zero (otherwise μ would blow up). The vanishing of μ also relates the constant part of μ and the coefficient of the Σ^{-1} term. If $R > 0$, then

$$M \;=\; \frac{K}{\Sigma} \;-\; \frac{K}{R}\,. \tag{4.5}$$

The coefficient K contributes to the equivalent of the "tube" angular momentum J_T. The value of μ at infinity is:

$$\lim_{r\to\infty}\mu \;=\; -\frac{K}{R} \;-\; \frac{2}{3h^2}C_{IJK}\gamma^I\gamma^J\gamma^K \;+\; \frac{1}{h}c_I\gamma^I\,, \tag{4.6}$$

and its effect is to reduce the size of the compactification circle.

We could now begin finding the coefficients of the various terms appearing on the right hand side of (3.16) and building up ω from $v_{1,2}$, and c_{12} such that (3.16) is satisfied. We would then have to choose the integration constants and relations between the parameters of the solutions such that ω vanishes on the z axis.

However, it is easier to work backwards, and begin by recalling that the absence of CTC's implies that ω must be proportional to ω_0 defined in (3.26). We then use equation (3.16) to determine the proportionality coefficient in terms of the charges of the solutions

$$\omega \;=\; \frac{KQ}{R}\,\omega_0\;, \tag{4.7}$$

and to find the relation that determines the radius of the black ring as a function of the charges, dipole charges and moduli:

$$K\left(h+\frac{Q}{R}\right) = -2\,h^{-1}\,C_{IJK}\beta^I\gamma^J\gamma^K + c_I\beta^I - g_I\gamma^I\,. \tag{4.8}$$

If we fix the moduli to their value in Section 2 ($\gamma_i = 0,\; Q = 1$ and $c_i = 1$), and note that in that case:

$$\beta^I = -\frac{q^I}{4}\;,\quad g_I = \frac{\overline{Q}_I}{4}\;,\quad K = -\frac{J_T}{16}\,, \tag{4.9}$$

equation (4.8) reproduces the radius formula (2.11).

One can also compute the value of the entropy for the most generic choice of moduli, and obtain

$$S = 2\pi\sqrt{64\mathcal{J}} \tag{4.10}$$

where

$$\begin{aligned}\mathcal{J} \;&=\; -(\beta^I\,g_I)^2 + C^{IJK}\,C_{IMN}\,g_J\,g_K\,\beta^M\,\beta^N \;-\; \frac{8}{3}K\,C_{IJK}\,\beta^I\,\beta^J\,\beta^K \\ &=\; -\;\left(g_1\,\beta^1 + g_2\,\beta^2 + g_3\,\beta^3\right)^2 \;-\; 16K\,\beta^1\,\beta^2\,\beta^3 \\ &\quad +\; 4g_1\,\beta^1\,g_2\,\beta^2 \;+\; 4g_1\,\beta^1\,g_3\,\beta^3 \;+\; 4g_2\,\beta^2\,g_3\,\beta^3\,.\end{aligned} \tag{4.11}$$

The relation between β^I and the dipole charges cannot depend upon the moduli because they are determined by the periods of the field strengths taken over the two-cycles. Hence, if the relation between the M2 charges, $\overline{Q}_I$, and the parameters g_I, and between J_T and K, remains the one in equation (4.9) for arbitrary moduli, then one can explain the

entropy (4.10) in exactly the same way as before. However, it is also possible, although unlikely, that when the γ^I are non-zero the $\overline{Q}_I$ and J_T could be given by some other more complicated combinations of the parameters, such that the $E_{7(7)}$ form of (4.11) is preserved when written in terms of the $\overline{Q}_I$ and J_T. We leave the exploration of this and other issues to future work. Here we simply note that for the solutions that reduce to five-dimensional black ring solutions in the near-tip limit (like the one considered in section 2), all the γ^I vanish, and no such ambiguity exists.

5. Conclusions and Future Directions

We have presented the solution for a general black ring in Taub-NUT. The general solution carries six charges, angular momentum, and has three moduli corresponding to the sizes of two-cycles in T^6 and the Taub-NUT fibre, and three moduli corresponding to Wilson lines along the fibre. We have also identified a gauge invariance in the equations governing these rings, which can be used to show that our solution is the most general solution describing a BPS black ring in Taub-NUT.

When the ring is localized near the origin of Taub-NUT the solution reduces to the five-dimensional BPS black ring, while in the opposite limit, where the ring is far from the origin, we recover the four-dimensional black hole corresponding to a black string wrapped on a circle.

By being able to smoothly interpolate between these two descriptions we resolved some confusion regarding the microscopic description of black rings. In particular, in [11] the black ring entropy was accounted for in the $(4,0)$ CFT of a four-dimensional black hole with charges $\overline{Q}_I$. For the five-dimensional black rings $\overline{Q}_I$ are the "local" charges of the ring, and differ from the conserved charges Q_I measured at infinity. However, these charges are hard to define as flux integrals, and this led to some debate regarding the validity of the description of $\overline{Q}_I$ as a ring charge. Now this issue can be resolved unambiguously: by taking the black ring out to large radius in the Taub-NUT space we see explicitly that it turns into a four-dimensional black hole with charges $\overline{Q}_I$, and this makes it clear that the CFT description should be based on these charges. Furthermore, since the charges are quantized this conclusion continues to hold even when we bring the ring back near the origin, where it becomes effectively five-dimensional.

It is straightforward to generalize our solution to the case of multiple rings, along the lines of [20,6]. One simply adds additional source points in the harmonic functions. The CFT description is just that of a collection of four-dimensional black hole CFT's, each with the corresponding charge $\overline{Q}_I$. The key point is that it is the $\overline{Q}_I$ and the q^I which *are additive*, while the $Q_I = \overline{Q}_I + \frac{1}{2}C_{IJK}q^J q^K$ *are not*. This, together with the fact that the entropy of separated rings is additive implies that any putative CFT description based on the Q_I will need to have a mechanism for disentangling the contributions of each individual ring, which seems rather difficult to achieve.

As a final comment we should note that in the type IIB frame the black rings have a near horizon limit which is asymptotically $\mathrm{AdS}_3 \times S^3$. Therefore, the black rings exist as states in the same D1-D5 CFT as the usual five-dimensional black hole. A microscopic description of the black rings in this CFT was proposed in [11]. However, this proposal rested on one phenomenological assumption regarding the length of effective strings in the CFT. Perhaps this assumption can be established more firmly using the ideas developed here.

Acknowledgments:

The work of IB and PK is supported in part by the NSF grant PHY-00-99590. The work of NW is supported in part by the DOE grant DE-FG03-84ER-40168.

References

[1] I. Bena and P. Kraus, "Three charge supertubes and black hole hair," Phys. Rev. D **70**, 046003 (2004) [arXiv:hep-th/0402144].

[2] I. Bena, "Splitting hairs of the three charge black hole," Phys. Rev. D **70**, 105018 (2004) [arXiv:hep-th/0404073].

[3] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, "A supersymmetric black ring," Phys. Rev. Lett. **93**, 211302 (2004) [arXiv:hep-th/0407065].

[4] I. Bena and N. P. Warner, "One Ring to Rule Them All ... and in the Darkness Bind Them?," arXiv:hep-th/0408106.

[5] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, "Supersymmetric black rings and three-charge supertubes," Phys. Rev. D **71**, 024033 (2005) [arXiv:hep-th/0408120].

[6] J. P. Gauntlett and J. B. Gutowski, "General concentric black rings," Phys. Rev. D **71**, 045002 (2005) [arXiv:hep-th/0408122].

[7] P. Kraus and F. Larsen, "Attractors and black rings," arXiv:hep-th/0503219.

[8] I. Bena and P. Kraus, "Microstates of the D1-D5-KK system," arXiv:hep-th/0503053.

[9] F. Denef, "Supergravity flows and D-brane stability," JHEP **0008**, 050 (2000) [arXiv:hep-th/0005049]. B. Bates and F. Denef, "Exact solutions for supersymmetric stationary black hole composites," arXiv:hep-th/0304094. F. Denef, "Quantum quivers and Hall/hole halos," JHEP **0210**, 023 (2002) [arXiv:hep-th/0206072].

[10] J. B. Gutowski and H. S. Reall, "General supersymmetric AdS(5) black holes," JHEP **0404**, 048 (2004) [arXiv:hep-th/0401129].

[11] I. Bena and P. Kraus, "Microscopic description of black rings in AdS/CFT," JHEP **0412**, 070 (2004) [arXiv:hep-th/0408186].

[12] M. Cyrier, M. Guica, D. Mateos and A. Strominger, "Microscopic entropy of the black ring," arXiv:hep-th/0411187.

[13] G. T. Horowitz and H. S. Reall, "How hairy can a black ring be?," Class. Quant. Grav. **22**, 1289 (2005) [arXiv:hep-th/0411268].

[14] D. Gaiotto, A. Strominger and X. Yin, "New connections between 4D and 5D black holes," arXiv:hep-th/0503217.

[15] H. Elvang, R. Emparan, D. Mateos and H. S. Reall, "Supersymmetric 4D rotating black holes from 5D black rings," arXiv:hep-th/0504125.

[16] D. Gaiotto, A. Strominger and X. Yin, "5D Black Rings and 4D Black Holes," arXiv:hep-th/0504126.

[17] R. Kallosh and B. Kol, "E(7) Symmetric Area of the Black Hole Horizon," Phys. Rev. D **53**, 5344 (1996) [arXiv:hep-th/9602014].

[18] M. Bertolini and M. Trigiante, "Microscopic entropy of the most general four-dimensional BPS black hole," JHEP **0010**, 002 (2000) [arXiv:hep-th/0008201].

[19] J. M. Maldacena, A. Strominger and E. Witten, “Black hole entropy in M-theory,” JHEP **9712**, 002 (1997) [arXiv:hep-th/9711053].

[20] J. P. Gauntlett and J. B. Gutowski, “Concentric black rings,” Phys. Rev. D **71**, 025013 (2005) [arXiv:hep-th/0408010].

[21] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, “All supersymmetric solutions of minimal supergravity in five dimensions,” Class. Quant. Grav. **20**, 4587 (2003) [arXiv:hep-th/0209114].